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Semiregular automorphisms of arc-transitive graphs with valency pq ☆

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Abstract

It was conjectured (see [D. Marušič, On vertex symmetric digraphs, Discrete Math. 36 (1981) 69–81]) that every vertex-transitive digraph has a semiregular automorphism, that is, a nonidentity automorphism having all orbits of equal length. Despite several partial results supporting its content, the conjecture remains open. In this paper, it is shown that the conjecture holds whenever the graph is arc-transitive of valency pq, where p and q are primes (p may equal q), and such that its automorphism group has a nonabelian minimal normal subgroup with at least three orbits on the vertex set.

1. Introduction

There has recently been increased interest in the so-called polycirculant conjecture. This conjecture originates from the study of vertex-transitive graphs. A *graph* is a pair $\Gamma = (\Omega, E)$ where Ω is the set of *vertices* of Γ, and E is a subset of unordered pairs of distinct vertices which are called the *edges* of Γ; the ordered pairs (α, β) of adjacent vertices are called the *arcs* of Γ. A *vertex-transitive graph* is a graph whose automorphism group acts transitively on the set of vertices. Moreover, a vertex-transitive graph Γ is said to be *arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of arcs of Γ.

In 1981, Marušič [10] asked if every finite vertex-transitive graph has a semiregular automorphism. A *semiregular* permutation is a non-identity permutation whose cycles all have the same length. Such a permutation generates a semiregular permutation group, that is, a group

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such that the only element which fixes a point is the identity. Note that a vertex-transitive graph is a Cayley graph if and only if it has a regular group of automorphisms. Hence Marušič's question can be viewed as a natural generalization of this proposition of Cayley graphs.

So far, it was proved that every vertex-transitive graph of valency three [11] or four [5] has a nonidentity semiregular automorphism. Moreover all vertex-transitive digraphs with $2p^2$, p^k or mp vertices, where p is a prime and $m \leq p$, have a semiregular automorphism of order p (see [10,11]). It was shown in [4] that every vertex-transitive graph of square-free order has a semiregular automorphism. It has recently been proved (see [9]) that all vertex-transitive locally quasiprimitive graphs have a nonidentity semiregular automorphism. We say that a permutation group is *quasiprimitive* if every nontrivial normal subgroup is transitive, and that a graph Γ with a group of automorphisms G is G*-locally quasiprimitive* if for each vertex α, the vertex stabilizer G_α acts quasiprimitively on the set $\Gamma(\alpha)$ of vertices adjacent to α.

Recall that the 2-*closure* (see [14]) $G^{(2)}$ of a permutation group G is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same set of orbits on ordered pairs as G. We say that G is 2-*closed* if it is equal to its 2-closure. Noting that the full automorphism group of a graph is 2-closed, Klin [2, Problem BCC15.12] extended the question of Marušič to the more general setting of 2-closed groups. This leads to what is now known as the *polycirculant conjecture* (see [1]), that every finite transitive 2-closed permutation group of degree at least two contains a nontrivial semiregular permutation. The name 'polycirculant' suggests that the vertex-transitive digraph with a semiregular automorphism contains several circulants of equal order and so has nice representations. On the other hand, we say that a permutation group $G \leq \mathrm{Sym}(\Omega)$ is *elusive* if G is transitive and contains no nontrivial semiregular permutations (equivalently, no fixed point free elements of prime order). For instance, the Mathieu group M_{11} in its 3-transitive action on 12 points is elusive. See [1,8] for more examples. Clearly, the study of elusive permutation groups is closely related to the study of the polycirculant conjecture. Giudici [7] determined all elusive permutation groups which has a transitive minimal normal subgroup, and showed that none of them are 2-closed. A *biquasiprimitive* permutation group is a transitive permutation group for which every nontrivial normal subgroup has at most two orbits and there is some normal subgroup with precisely two orbits. Such groups arise naturally when studying bipartite graphs, and in [9] all elusive biquasiprimitive permutation groups have been determined and shown not to be 2-closed.

Following the work of [9], it is natural to ask if the polycirculant conjecture is true for arc-transitive graphs which are not necessarily locally quasiprimitive. In this paper, we take the first step to look at arc-transitive graphs Γ with valency pq where p, q are two primes, (note that p may be equal to q). Let $G = \mathrm{Aut}(\Gamma)$. It follows from [7, Theorem 1.2] and [9, Theorem 1.4] that G contains a nonidentity semiregular element if G is either quasiprimitive or biquasiprimitive. Therefore we may suppose that there exists a minimal normal subgroup M of G such that M has at least 3 orbits on the vertex set. In this paper we concentrate on the case that this M is nonabelian. Our main result is Theorem 1.1. To prove this result, we investigate the elusive permutation groups which have an intransitive nonabelian minimal normal subgroup in Section 3.

Theorem 1.1. *Let Γ be a finite connected arc-transitive graph with valency pq where p and q are primes. Suppose further that* $\mathrm{Aut}(\Gamma)$ *has a nonabelian minimal normal subgroup M such that M has at least 3 orbits on the vertex set. Then Γ has a nonidentity semiregular automorphism.*

Finally, we give a few words about the case where the above M is abelian. So far, we are still working on this case. It turns out that this abelian case is rather difficult and requires very different methods.

2. Preliminary results

First we introduce notation and basic definitions. Let $\Gamma = (\Omega, E)$ be a connected arc-transitive graph. For $\alpha \in \Omega$, we use $\Gamma(\alpha)$ to denote the *neighborhood* of α in Γ, that is, the set of vertices of Γ adjacent to α. For a subset X of Ω, the subgraph of Γ *induced* by X is the graph with vertex set X in which $\alpha, \beta \in X$ are adjacent if and only if they are adjacent in Γ.

The fundamental tool for studying imprimitive arc-transitive graphs is analysis of the normal quotient graph, which was introduced by Praeger in [12]. Let $G = \mathrm{Aut}(\Gamma)$, and let N be an intransitive normal subgroup of G. Then the *normal quotient graph* Γ_N is defined as the graph with vertices the N-orbits such that two N-orbits Δ_1 and Δ_2 are adjacent in Γ_N if and only if some $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$ are adjacent in Γ. Then it is easy to see that G induces an arc-transitive action on this quotient graph as well.

Next we collect some known results about elusive permutation groups. Recall that the 3-transitive action of M_{11} on 12 points with point stabilizer $\mathrm{PSL}(2, 11)$ is elusive. The following lemma gives more examples, its proof can be found in [1,7].

Lemma 2.1. *Let $G = M_{11} \wr K$ acting with its product action on $\Omega = \Delta^k$ for some $k \geq 1$, where K is a subgroup of S_k and $|\Delta| = 12$. Then the following hold.*

1. *G is elusive on Ω.*
2. *$G^{(2)}$ contains the subgroup $S_{12} \wr K$, and hence is not elusive.*
3. *M_{11}^k is elusive on Ω and $(M_{11}^k)^{(2)} = S_{12}^k$ is not elusive.*

Giudici [7] has proved the following results.

Theorem 2.2 (*[7, Theorem 1.1]*). *Let G be an elusive permutation group on a finite set Ω which has at least one transitive minimal normal subgroup. Then $G = M_{11} \wr K$ acting with its product action on $\Omega = \Delta^k$ for some $k \geq 1$, where K is transitive subgroup of S_k and $|\Delta| = 12$.*

Theorem 2.3 (*[7, Theorem 1.3]*). *Let T be a simple group with a proper subgroup R which meets every* $\mathrm{Aut}(T)$*-conjugacy class of elements of T of prime order. Then T is one of $A_6, M_{11}, P\Omega^+(8, 2)$ or $P\Omega^+(8, 3)$, and $R = A_5, \mathrm{PSL}(2, 11), P\Omega(7, 2)$ or $P\Omega(7, 3)$, respectively. Furthermore, if R meets every conjugacy class of elements of T of prime order, then $T = M_{11}$ and $R =$* $\mathrm{PSL}(2, 11)$.

In Section 3, we will continue our investigation on elusive permutation groups.

Finally, we quote the following well known result concerning transitive permutation groups of prime degree.

Proposition 2.4 (*[3, Section 3.5]*). *Let G be a transitive permutation group of prime degree p. Then G is one of the following.*

1. *The symmetric group S_p or the alternating group A_p.*
2. *A subgroup of* $\mathrm{AGL}(1, p)$.
3. *A permutation representation of* $\mathrm{PSL}(2, 11)$ *of degree* 11.
4. *One of the Mathieu groups M_{11} or M_{23} of degree* 11 *or* 23*, respectively.*
5. *A projective group G with* $\mathrm{PSL}(d, q) \leqslant G \leqslant P\Gamma L(d, q)$ *of degree $p = (q^d - 1)/(q - 1)$.*

3. Elusive permutation groups

Throughout this section we let G be an elusive permutation group on Ω. Suppose also that $N = T^k$ is an intransitive nonabelian minimal normal subgroup of G. We will investigate the action of N on each of its orbits in general.

Let $\Delta_1, \ldots, \Delta_s (s \geq 2)$ be the orbits of N on Ω. Let T_j be the jth simple direct factor of N and write $N = T_1 \times \cdots \times T_k$ where each $T_j \cong T$. For each $j \in \{1, \ldots, k\}$, let

$$\Pi_j : N \to T_j$$

denote the projection onto the jth simple direct factor of N. Let $\alpha \in \Delta_1$ and let N_α be the stabilizer of α in N. We consider N_α.

The proof of the following lemma can be found in [6, Lemma 4.5.1] and [9, Lemma 4.3].

Lemma 3.1. *Let G be an elusive group on a set Ω and let N be a normal subgroup of G such that $N \cong T^k$ for some nonabelian simple group T. Let $\alpha \in \Omega$. Then for each $j \in \{1, \ldots, k\}$, the projection $\Pi_j(N_\alpha)$ either equals T_j or is a proper subgroup of T_j which meets every* $\mathrm{Aut}(T_j)$*-conjugacy class of elements of T_j of prime order.*

Giudici has observed the following facts in his thesis [6].

Remark 3.2 (*See [6, page 59]*). With above notation, Lemma 3.1 and Theorem 2.3 enable us to write N_α as follows. Without loss of generality, we may suppose that there exists a nonnegative integer c, possibly 0, such that for $j \in \{1, \ldots, c\}$, the projection $\Pi_j(N_\alpha) = T$ and that T is a direct factor. Note that T^c is the kernel of the action of N on the orbit containing α, so $c < k$ and if $c = 0$, then N is faithful on each of its orbits. Furthermore, we may suppose that there is some $m \geq c$, such that for $j \in \{c + 1, \ldots, m\}$ the projection $\Pi_j(N_\alpha) = R$, a proper subgroup of T, and R is a direct factor. We can then write

$$N_\alpha = T^c \times R^{m-c} \times H'$$

where $H' \leqslant T^{k-m}$. If $m \neq k$, then we will suppose that there is some integer n such that for $j \in \{m + 1, \ldots, n\}$, the projection $\Pi_j(N_\alpha) = T$ and for $j \in \{n + 1, \ldots, k\}$ the projection $\Pi_j(N_\alpha) = R$ and that none of these are direct factors. Then we have that $\Pi_{\{m+1,\ldots,n\}}(N_\alpha)$ is a subdirect product of T^{n-m} and so by [13, page 328], is a product of diagonals. Similarly, $\Pi_{\{n+1,\ldots,k\}}(N_\alpha)$ is a subdirect product of R^{k-n}. Since R is simple by Theorem 2.3, [13, page 328] again implies that $\Pi_{\{n+1,\ldots,k\}}(N_\alpha)$ is also a product of diagonals.

With this representation of N_α in mind, we have the following lemma.

Lemma 3.3. *Let $N = T^k$ be an intransitive normal subgroup of an elusive group G on a set Ω such that T is nonabelian simple. Let $\alpha \in \Delta_1$ where Δ_1 is an orbit of N on Ω. Suppose that for each $j \in \{1, \ldots, k\}$, the projection $\Pi_j(N_\alpha)$ equals T_j. Then for each prime $r \mid |T|$, there exists $x \in N$ with order r, such that x fixes no points on Δ_1.*

Proof. Using the notation of Remark 3.2, we have $N_\alpha = T^c \times \prod_{I \in \mathcal{P}} D_I$ where $c < k$, $\mathcal{P}$ is a partition of $\{c + 1, \ldots, k\}$ and D_I are diagonal subgroups. For each prime $r \mid |T|$, let $x = (1, \ldots, 1, t) \in N$ be an element of order r. Since N^{Δ_1} is transitive and $x \notin N_\alpha^g$ for each $g \in N$, we have that x cannot fix any point on Δ_1. $\square$

Lemma 3.4. *Let $N = T^k$ be an intransitive normal subgroup of an elusive group G on a set Ω such that T is nonabelian simple. Let $\alpha \in \Delta_1$ where Δ_1 is an orbit of N on Ω. Suppose that there exists at least one $i \in \{1, \ldots, k\}$ such that $\Pi_i(N_\alpha)$ is a proper subgroup of T_i, then the following results hold.*

1. *$T = A_6, M_{11}, P\Omega^+(8, 2)$ or $P\Omega^+(8, 3)$, and $\Pi_i(N_\alpha) \cong A_5, \mathrm{PSL}(2, 11), P\Omega(7, 2)$ or $P\Omega(7, 3)$, respectively.*
2. *With the notation of Remark 3.2, suppose further $k > m$, that is for each $j \in \{m + 1, \ldots, k\}$, $\Pi_j(N_\alpha)$ is not a direct factor. Then for each prime $r \mid |T|$, there exists $x \in N$ with order r, such that x fixes no points on Δ_1.*
3. *Suppose further that $N = A_6^k$ and $N_\alpha = A_6^c \times A_5^{k-c}$. Then $N^{\Delta_1} = A_6^{k-c}$ and $(N^{\Delta_1})_\alpha = A_5^{k-c}$, a natural product action. Moreover there exists $x \in N$ with order 3 such that x fixes no points on Δ_1.*
4. *Suppose further that $N = M_{11}^k$ and $N_\alpha = \mathrm{PSL}(2, 11)^k$. Then the 2-closure $N^{(2)}$ contains a fixed point free element of order 3 on Ω.*

Remark. The proof of 4 is essentially the same as the proof of [9, Lemma 3.2].

Proof. 1. It follows from Theorem 2.3 and Lemma 3.1.

2. For each prime $r \mid |T|$, let $x = (1, \ldots, 1, t) \in N$ be an element of order r. The same argument as in the proof of Lemma 3.3 implies that x fixes no points on Δ_1.

3. By assumption, we have that N^{Δ_1} is a natural product action. Then we may suppose that $\Delta_1 = \Phi^{k-c}$ where $\Phi = [A_6 : A_5]$ is the set of cosets of A_5 in A_6 and $|\Phi| = 6$. Let $t \in A_6$ with cycle structure 3^2, and let $x = (1, \ldots, 1, t) \in N = A_6^k$. Then for each $g \in N$, $x \notin N_\alpha^g$. Thus x fixes no points on Δ_1 as required.

4. By assumption and Lemma 2.1, we know that N is faithful and elusive on each Δ_i. By the embedding theorem for permutation groups, N is conjugate in $\mathrm{Sym}(\Omega)$ to a subgroup $\mathrm{Diag}(H \times \cdots \times H)$ where $H = N^{\Delta_1} \cong M_{11}^k$ and there exists $\varphi_2, \ldots, \varphi_s \in \mathrm{Aut}(H) = M_{11} \wr S_k \leq S_{12} \wr S_k$ such that

$$\mathrm{Diag}(H \times \cdots \times H) = \{(a, a^{\varphi_2}, a^{\varphi_3}, \ldots, a^{\varphi_s}) | a \in H\}.$$

Denote $\Omega = \Delta \times \{1, \ldots, s\}$, that is, $\Delta_i = \Delta \times \{i\}$, and let $\varphi_1 = id$. Then N acts on Ω via $(\delta, i)^{(a, a^{\varphi_2}, \ldots, a^{\varphi_s})} = (\delta^{a^{\varphi_i}}, i)$ for each i.

The orbits of $N = \mathrm{Diag}(H \times \cdots \times H)$ on the set $\Omega \times \Omega$ are of the following forms.

1. $\{((\alpha, i), (\beta, i)) : (\alpha, \beta) \in \mathcal{O}\}$, for each i and each orbit $\mathcal{O}$ of H on $\Delta_1 \times \Delta_1$.
2. $\{((\alpha, i), (\beta, j)) : i \neq j, (\alpha, \beta) \in \mathcal{O}^{ij}_{\gamma,\delta}\}$, where $\mathcal{O}^{ij}_{\gamma,\delta} = \{(\gamma^{g^{\varphi_i}}, \delta^{g^{\varphi_j}}) : g \in H\}$ for some $\gamma, \delta \in \Delta_1$.

Note that $H = N^{\Delta_1}$ is the product action, we may write $\Delta = \Phi^k$ and $|\Phi| = 12$. Let $y \in S_{12}$ be a fixed point free element of order 3 on Φ. Then the same argument as in the proof of [9, Lemma 3.2] will prove that $h = ((y, 1, \ldots, 1), (y, 1, \ldots, 1)^{\varphi_2}, \ldots, (y, 1, \ldots, 1)^{\varphi_s})$ acting on $\Delta \times \{1, \ldots, s\}$ preserves all N-orbits on $\Omega \times \Omega$, and hence h is a fixed point free element of order 3 in $N^{(2)}$. $\square$

4. Normal quotient graphs

Throughout this section we let $\Gamma = (\Omega, E)$ be a finite connected arc-transitive graph with valency pq where p and q are primes (p may equal q). Let $G = \mathrm{Aut}(\Gamma)$ and let $M = T^k$ be a nonabelian minimal normal subgroup of G such that M has at least 3 orbits on Ω. We begin our investigation by looking at the normal quotient graph Γ_M.

Lemma 4.1. *With above notation, let $\alpha \in \Omega$. Suppose that M contains no fixed point free elements of prime order on Ω. Then the length of the orbits of M_α on $\Gamma(\alpha)$ is p (or q). Furthermore, p (or q) is the maximal prime divisor of $|T|$.*

Proof. Since $M_\alpha \trianglelefteq G_\alpha$ and $G_\alpha^{\Gamma(\alpha)} \leq S_{pq}$ is a transitive permutation group of degree pq, we have that the length l of M_α-orbits on $\Gamma(\alpha)$ is 1, p, q or pq.

Suppose first $l = 1$. Let $g \in M_\alpha$. Since Γ is connected and $M \trianglelefteq G$, g fixes Ω pointwise. Thus $M_\alpha = 1$ and M is semiregular, contradicting our hypothesis. Thus $l > 1$.

Secondly, since M has at least 3 orbits, we have $l \neq pq$. Otherwise, M is intransitive implies that M has only two orbits on Ω as Γ is connected. Therefore, we have proved that $l = p$ (or q).

Finally, suppose there exists a prime $r > p$ with $r \mid |T|$. Pick an element $g \in M$ with $o(g) = r$. Suppose $g \in M_\beta$ for some vertex β. Then g fixes $\Gamma(\beta)$ pointwise as $o(g) > p = l$. Thus g fixes Ω pointwise as Γ is connected, which is not the case. Hence g is fixed point free, contradicting our hypothesis again. Therefore the maximal prime divisor of $|T|$ is p (or q). □

Let $\Delta_1, \ldots, \Delta_s$ be the orbits of M on Ω where $s \geq 3$. If (Δ_i, Δ_j) is an edge of Γ_M, then the subgraph of Γ induced by Δ_i and Δ_j is denoted by $\Sigma_{ij} := [\Delta_i, \Delta_j]$.

Corollary 4.2. *With the above notation, suppose that G is elusive. Then the normal quotient graph Γ_M is an arc-transitive graph of prime valency q (or p), and the induced subgraph $\Sigma_{ij} := [\Delta_i, \Delta_j]$ is a bipartite graph of valency p (or q) if (Δ_i, Δ_j) is an edge of Γ_M.*

Proof. Since G is elusive, M contains no fixed point free elements on Ω. By Lemma 4.1, the length of M_α-orbits on $\Gamma(\alpha)$ is p (or q). The result then follows easily. □

Without loss of generality, we may suppose that the normal quotient graph Γ_M is an arc-transitive graph of valency q, that is, the length of the orbits of M_α on $\Gamma(\alpha)$ is p.

Lemma 4.3. *With the above notation, suppose that the normal quotient graph Γ_M is an arc-transitive graph of valency q. Then the following hold.*

1. *If there exists $x \in M$ with prime order $r \neq p$ such that x fixes no points on some M-orbit Δ_m, then x is fixed point free on Ω.*
2. *Suppose M is unfaithful on Δ_1. Then $\Sigma_{ij} := [\Delta_i, \Delta_j]$ is isomorphic to $n \cdot K_{p,p}$ where (Δ_i, Δ_j) is an edge of Γ_M and n is a positive integer. In particular, $p \mid |\Delta_1|$.*

Proof. 1. If (Δ_m, Δ_i) is an edge in Γ_M, then x fixes no points on Δ_i as $\Sigma_{mi} := [\Delta_m, \Delta_i]$ is a bipartite graph of valency p and $p \neq r$. Since Γ is connected, x is a fixed point free element of order r on Ω.

2. Let K_i be the kernel of the action of M on Δ_i for each i. Since M^{Δ_i} is permutationally isomorphic to M^{Δ_1}, $K_i \cong K_1 \cong T^c$ where $1 \leq c < k$. Note that $p \mid |T|$ as $p \mid |M_\alpha|$. Since Γ is connected, it is easy to see that there exists an edge, say (Δ_1, Δ_2) in Γ_M, such that we can pick $g_1 \in K_1 \setminus K_2$ with $o(g_1) = p$.

Next we look at the subgraph $\Sigma_{12} = [\Delta_1, \Delta_2]$ induced by Δ_1 and Δ_2. Suppose $(\gamma_1, \gamma_2, \ldots, \gamma_p)$ is a cycle of $g_1^{\Delta_2}$. Suppose also that $\Gamma(\gamma_1) \cap \Delta_1 = \{\delta_1, \ldots, \delta_p\}$. Since $g_1 \in K_1$, the subgraph induced by $\{\delta_1, \ldots, \delta_p\}$ and $\{\gamma_1, \ldots, \gamma_p\}$ is isomorphic to $K_{p,p}$. Note that M is transitive on Δ_1 and Δ_2, hence $[\Delta_1, \Delta_2] \cong n \cdot K_{p,p}$ for some integer n. Since G induces an arc-transitive action on Γ_M, for each edge (Δ_i, Δ_j), we have $\Sigma_{ij} := [\Delta_i, \Delta_j]$ is isomorphic to $n \cdot K_{p,p}$. In particular, $p \mid |\Delta_1|$ as asserted. □

5. Proof of Theorem 1.1

We are ready to prove our main theorem.

Proof of Theorem 1.1. Suppose that $\Gamma = (\Omega, E)$ is a connected arc-transitive graph with valency pq where p, q are two primes. Let $G = \mathrm{Aut}(\Gamma)$. Suppose that M is a nonabelian minimal normal subgroup of G such that M has at least 3 orbits on Ω.

Let $M = T_1 \times \cdots \times T_k \cong T^k$ where $k \geq 1$ and for each $i \in \{1, \ldots, k\}$, $T_i \cong T$ is a nonabelian simple group.

Let $\Delta_1, \ldots, \Delta_s$, $s \geq 3$, be orbits of M on Ω. Then M^{Δ_1} is permutationally isomorphic to M^{Δ_i} for each $i \in \{1, \ldots, s\}$.

Suppose that G is elusive. It follows from Lemma 4.1 and Corollary 4.2 that the normal quotient graph Γ_M is an arc-transitive graph of valency p or q. Without loss of generality, we suppose that the valency of Γ_M is q. Then p is the maximal prime divisor of $|T|$ by Lemma 4.1.

Let $\alpha \in \Delta_1$. By Lemma 3.1, for each $j \in \{1, \ldots, k\}$, the projection $\Pi_j(M_\alpha)$ (into the jth simple direct factor T_j) either equals T_j or is a proper subgroup of T_j. We first suppose that for each $j \in \{1, \ldots, k\}$, the projection $\Pi_j(M_\alpha)$ equals T_j. Pick a prime divisor r of $|T|$ such that $r < p$. Then Lemma 3.3 implies that there exists $x \in M$ with order r, such that x fixes no points on Δ_1. Thus by Lemma 4.3(1), x is a fixed point free element of order r on Ω, contradicting our assumption that G is elusive. Therefore there exists at least one $i \in \{1, \ldots, k\}$ such that $\Pi_i(M_\alpha)$ is a proper subgroup of T_i. Lemma 3.4(1)(2) and Lemma 4.3(1) imply that $M_\alpha = T^c \times R^{k-c}$ where (T, R) is one of (A_6, A_5), $(M_{11}, \mathrm{PSL}(2, 11))$, $(P\Omega^+(8, 2), P\Omega(7, 2))$ or $(P\Omega^+(8, 3), P\Omega(7, 3))$. Note that M_α has a transitive permutation representation of degree p by Lemma 4.1. Thus Proposition 2.4 tells us that (T, R) is (A_6, A_5) or $(M_{11}, \mathrm{PSL}(2, 11))$, and p is 5 or 11 respectively. Now Lemma 3.4(3) and Lemma 4.3(1) imply that $(T, R) = (M_{11}, \mathrm{PSL}(2, 11))$, $p = 11$, and $|\Delta_1| = 12^{k-c}$. In particular, 11 does not divide $|\Delta_1|$, and so M is faithful on Δ_1 by Lemma 4.3(2), that is, $M_\alpha = \mathrm{PSL}(2, 11)^k$. It then follows from Lemma 3.4(4) that $M^{(2)}$ contains a fixed point free element of order 3 on Ω. Since $M^{(2)} \leqslant \mathrm{Aut}(\Gamma)^{(2)} = \mathrm{Aut}(\Gamma) = G$, G contains a fixed point free element of order 3, contradicting our assumption that G is elusive again.

Therefore G is not elusive, and hence Γ has a nonidentity semiregular automorphism. □

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